Analysis I

Lecture 4

Last time:

- Bounds of sets.
- MIN / MAX
- Inf / Sup

Today:

- Archemidian property
- Density of $\mathbb{Q} \subset \mathbb{R}$

Next week: Complex numbers

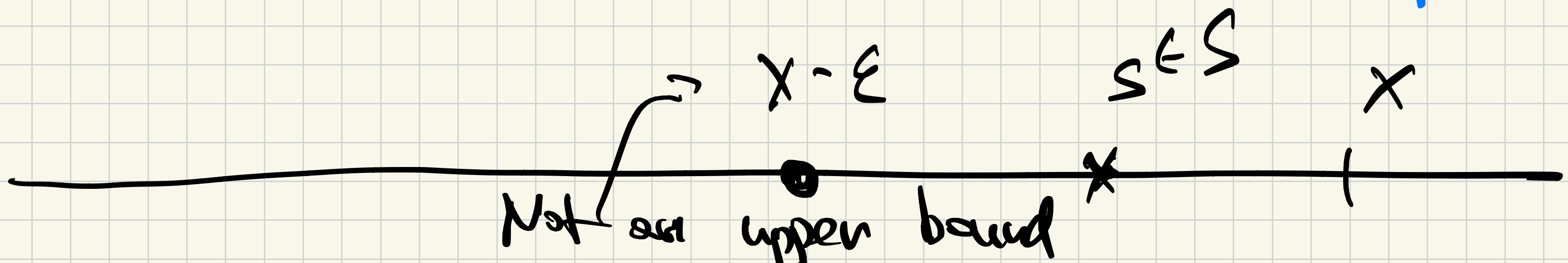
Set	Inf	Sup	Min	Max
(a, b)	a	b	No	No
$[a, b)$	a	b	a	No
$(a, b]$	a	b	No	b
$[a, b]$	a	b	a	b
$(a, +\infty)$	a	$+\infty$	No	No
$[a, +\infty)$	a	$+\infty$	a	No
$(-\infty, b)$	$-\infty$	b	No	No
$(-\infty, b]$	$-\infty$	b	No	b
$(-\infty, +\infty) = \mathbb{R}$	$-\infty$	$+\infty$	No	No

	Set	inf	sup	min	max
bounded	(a, b)	a	b	No	No
	$[a, b)$	a	b	a	No
	$(a, b]$	a	b	No	b
	$[a, b]$	a	b	a	b
bounded below	$(a, +\infty)$	a	$+\infty$	No	No
	$[a, +\infty)$	a	$+\infty$	a	No
bounded above	$(-\infty, b)$	$-\infty$	b	No	No
	$(-\infty, b]$	$-\infty$	b	No	b
	$(-\infty, +\infty) = \mathbb{R}$	$-\infty$	$+\infty$	No	No

Theorem Let $\emptyset \neq S \subset \mathbb{R}$, then $x \in \mathbb{R}$ is the supremum of S if and only if $x \geq s \ \forall s \in S$ and $\forall \varepsilon > 0 \ \exists s \in S$ such that $s > x - \varepsilon$.

x is an upper bound

Any number smaller then x is not an upper bound



Example $S = \left\{ \frac{n-1}{n} \,\middle|\, n \in \mathbb{N}^* \right\}$

$\inf(S) = 0$ Indeed, $\frac{n-1}{n} \geq 0$

so 0 is a lower bound.

But $n=1 \Rightarrow \frac{n-1}{n} = \frac{1-1}{1} = 0$

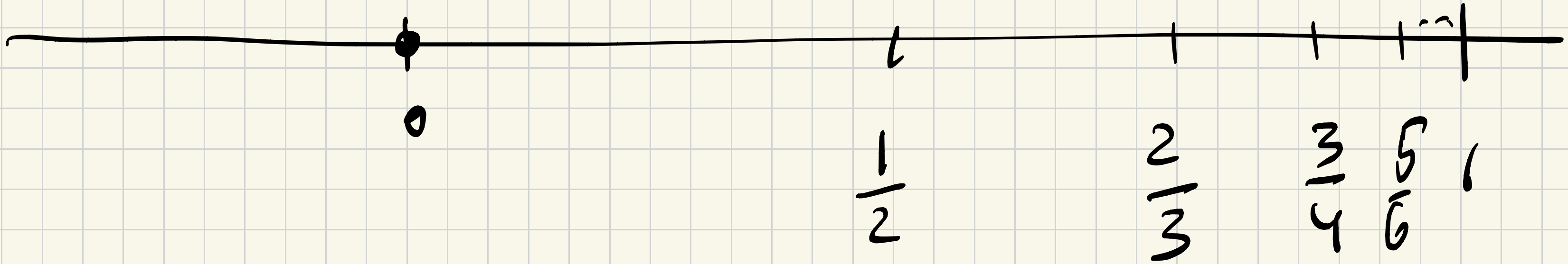
so $0 \in S \Rightarrow$

No positive number is a lower bound of S.

So we get $\inf(S) = \min(S) = 0$.

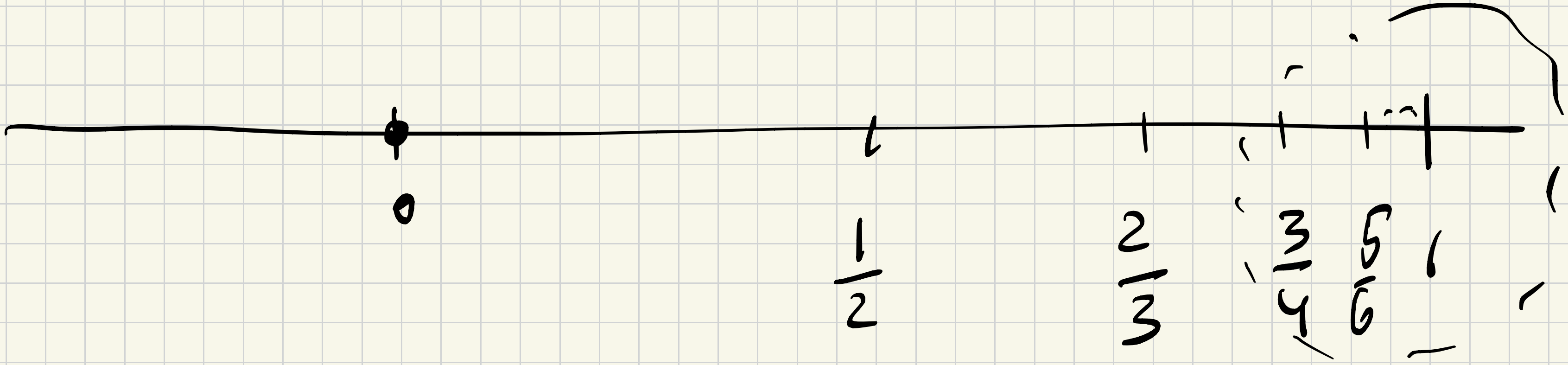
Now let's look for supremum;

$$\frac{n-1}{n} = 1 - \frac{1}{n} < 1 \quad \text{since } \frac{1}{n} > 0$$



$\Rightarrow$ 1 is an upper bound and

Also $1 \notin S$

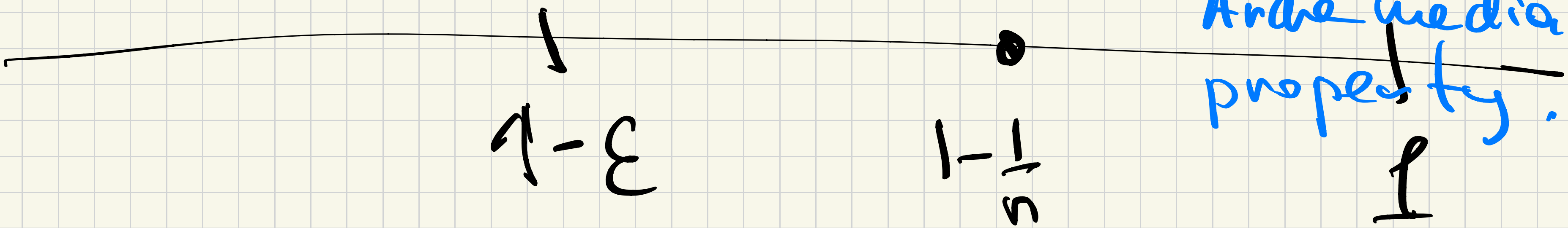


Intuitively if any $\varepsilon > 0$

then $\exists$ a number $1 - \frac{1}{n} > \underline{1 - \varepsilon} < 1$

$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$

Such n exists by Archemedian property.



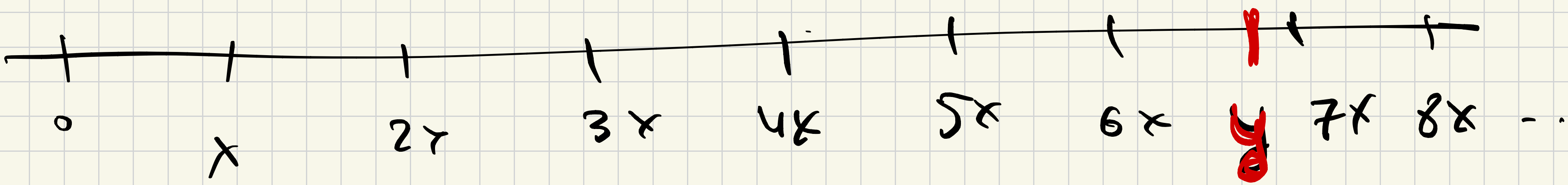
This implies that

$$\sup(S) = 1 \quad \text{But}$$

$\max(S)$ does not exist.

Theorem (ARCHEMIDEAN Property)

Let $x, y \in \mathbb{R}$ with $x > 0$ then there exists $n \in \mathbb{N}^*$ s.t. $nx > y$.



Example For any $y \in \mathbb{R}$ there exists $n \in \mathbb{N}^*$ s.t. $n > y$.

In this example $x = 1$, so

$$n \cdot x = n.$$

(In terms of decimal expression:

$y = 125237.621\ldots$ can take $n = 125237 + 1$)

Proof: We assumed $x > 0$, so

if $y \leq 0$ you can take $n = 1$

since $x > y$.

So we consider case when $y > 0$.

We will proceed by contradiction

Assume $y > n \cdot x \quad \forall n \in \mathbb{N}^*$

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Consider $S = \{ n \cdot x \mid n \in \mathbb{N}^* \}$

then S is bounded above by y.

Let $a = \sup(S) \in \mathbb{R}$

In particular a is an upper bound

$$n \cdot x \leq a \quad \forall\, n \in \mathbb{N}^*$$

In particular,

$$(n+1)\cdot x \le a \quad \forall n \in \mathbb{N}^*$$

$$\iff \quad nx + x \le a \quad \forall n \in \mathbb{N}^*$$

$$\iff \quad nx \le a - x < a \quad \forall n \in \mathbb{N}^*$$

But then $a - x$ is an upper bound for S $\Rightarrow$ a could not be the $\sup(S)$.

$\lightning$ contradiction.

Inf / Sup for subsets of $\mathbb{Z}$.

Proposition If $S \subset \mathbb{Z}$ is bounded below (above) then it has min (max).

In particular,

$$\inf(S) = \min(S) \in \mathbb{Z} \text{ if } S \text{ is bounded below}$$

$$\sup(S) = \max(S) \in \mathbb{Z} \text{ if } S \text{ is bounded above}$$

Proof. Proof by contradiction

Assume $\inf(S) \notin \mathbb{Z}$
$\quad\;\;\rotatebox{90}{=}$
$\quad\;\; a$

1st step: Take $a + \frac{1}{2} > a$ so it is not a lower bound

$\Rightarrow \exists\; s_1 \in S$ such that

$a \underset{\text{(!)}}{<} s_1 < a + \frac{1}{2}$

this is strict since $a \notin \mathbb{Z}$, $s_1 \in S \subset \mathbb{Z}$

2nd step $\quad s_1 > a \implies s_1$

$a < s_1$

is not a lower bound for S

$\Rightarrow \exists \; s_2 \in S$ with

$$a < s_2 < s_1$$

So we get $\underbrace{a < s_2 < s_1 < a + \frac{1}{2}}$

$\Rightarrow \underline{0} < s_1 - s_2 < \frac{1}{2}$ which is impossible since $s_1, s_2 \in \mathbb{Z}$

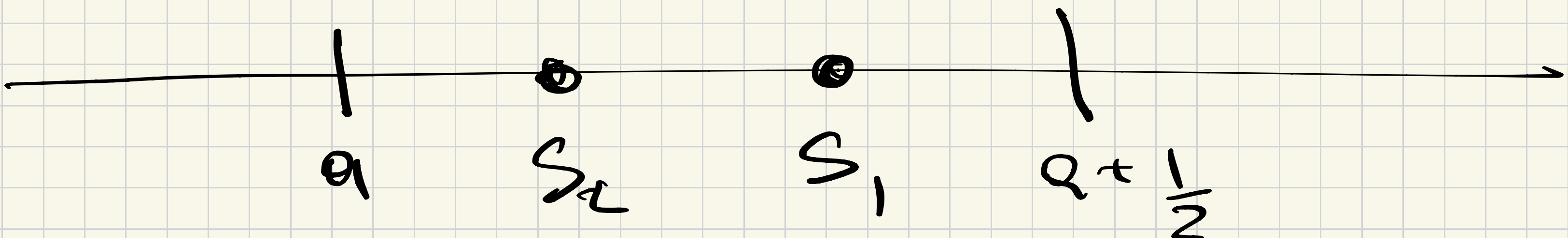
$0 < s_1 - s_2 < \frac{1}{2}$ which contradicts

the fact that $s_1, s_2 \in \mathbb{Z}$,

positive

so the smallest difference

is 1, $\lightning$



Integer parts:

Definition (Floor / Ceiling functions)

Let $x \in \mathbb{R}$ then

- the floor $\lfloor x \rfloor$ of x is the largest integer that is $\leq x$. Rounds down
- the ceiling $\lceil x \rceil$ of x is the smallest integer that is $\geq x$ Rounds up

- Integer part of x is

$$[x] := \begin{cases} \lfloor x \rfloor & \text{for } x \geq 0 \\ \lceil x \rceil & \text{for } x \leq 0 \end{cases}$$

Reason for this convention is:

$[45.0732\ldots] = 45$

$[-45.0732\ldots] = -45$

~~(Some people [x] = ⌊x⌋ but then [-45.0732...] = -46)~~

- Fractional part of x is

$$\{x\} := x - [x] \rightsquigarrow$$

$\{4.75\} = 0.75$

$\{-4.75\} = -0.75$

Example

If $x = 145.732$ — since $x > 0$

$\lfloor x \rfloor = 145$ $\quad \lceil x \rceil = 146$ $\quad [x] = \lfloor x \rfloor = 145$

$\{x\} = x - [x] = 145.732 - 145 = 0.732$

If $x = -145.732$ then — since $x < 0$

$\lfloor x \rfloor = -146$ $\quad \lceil x \rceil = -145$ $\quad [x] = \lceil x \rceil = -145$

$\{x\} = x - [x] = -145.732 - (-145) = -0.732$

Properties of integral/fractional parts

- $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$
- $\lceil x \rceil - 1 < x \le \lceil x \rceil$
- $[x] = -[-x]$
- $x = [x] + \{x\}$
- $x \in \mathbb{Z}$ iff $x = \lfloor x \rfloor = \lceil x \rceil = [x]$
- $-1 < \{x\} < 1$

Integral number
$\parallel$
Integer number

Exercise: prove these properties.

Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem Let $a, b \in \mathbb{R}$ with $a < b$

then there exists $c \in \mathbb{Q}$ s.t.

$a < c < b$.

Example define $S = \{ x \mid x \in \mathbb{Q} \text{ and } x^2 > 3 \}$

$\inf(S) = \sqrt{3}$

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to show this we relied on the fact that for any $a > \sqrt{3}$ there is a rational number $\sqrt{3} < c < a$.

Proof when $a = 0$

for any $b > 0$ $\exists$ rational number c

$$0 < c < b.$$

Indeed, consider $\underbrace{\left\lfloor \frac{1}{b} \right\rfloor + 1 > \frac{1}{b}}_{a}$

$a \in \mathbb{Z}_+$

$$\Rightarrow \quad 0 < \underbrace{\frac{1}{\left\lfloor \frac{1}{b} \right\rfloor + 1}}_{\in \mathbb{Q}} < b$$

c

$\blacksquare$